Sydney Girls High School

2021

ALTERNATE TASK 4

Mathematics Extension 2

General Instructions	• Reading time – 5 minutes • Working time – 60 minutes • Write using a black pen • Calculators approved by NESA may be used • A reference sheet is provided • In Questions 11-13, show relevant mathematical reasoning and/or calculations
Total marks: 46	**Section I – 10 marks (pages 1-5)** • Attempt Questions 1-10 • Allow about 13 minutes for this section **Section II – 36 marks (pages 6-8)** • Attempt Questions 11-13 • Allow about 47 minutes for this section

THIS IS NOT A TRIAL PAPER

It does not reflect the format or the content of the 2021 HSC Examination Paper in this subject.

Section I

10 marks
Attempt Questions 1–10

Write the question number and your response for Questions 1–10.

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1 Which expression is equal to $\displaystyle\int \frac{-1}{\sqrt{1-9x^2}}\,dx$?

A. $\dfrac{1}{3}\cos^{-1} 3x + C$

B. $\dfrac{1}{3}\cos^{-1} \dfrac{x}{3} + C$

C. $\cos^{-1} 3x + C$

D. $\cos^{-1} \dfrac{x}{3} + C$

2 Consider the following statement:

‘n is divisible by 4 and n is not divisible by 8’.

Which of the following is the negation of this statement?

A. n is not divisible by 4 or n is not divisible by 8.

B. n is not divisible by 4 and n is not divisible by 8.

C. n is not divisible by 4 or n is divisible by 8.

D. n is not divisible by 4 and n is divisible by 8.

3 The acceleration of a particle is given by $\ddot{x} = -\dfrac{1}{v}$.
What is the displacement of the particle as a function of v?

A. $x = \dfrac{1}{v} + C$

B. $x = -\dfrac{v^3}{3} + C$

C. $x = -\ln|v| + C$

D. $x = e^{-\frac{1}{2}v^2 + C}$

4 What is the value of the integral $\displaystyle\int_0^4 \frac{1}{x + \sqrt{x}}\,dx$ after making the substitution $u = \sqrt{x}$?

A. $\displaystyle\int_0^{16} \frac{2u}{u^2 + u}\,du$

B. $\displaystyle\int_0^{16} \frac{u}{2(u^2 + u)}\,du$

C. $\displaystyle\int_0^{2} \frac{2u}{u^2 + u}\,du$

D. $\displaystyle\int_0^{2} \frac{u}{2(u^2 + u)}\,du$

5 When $P(z) = 2z^3 + 3iz^2 + kz - 3i$ is divided by $z^2 + 1$, the remainder is $6z - 6i$.
What is the value of k?

A. $k = 5$

B. $k = 6$

C. $k = 7$

D. $k = 8$

6 If $\omega \neq 1$ is a cube root of unity, what is $(1+\omega^2)^3$ equal to?

A. 1

B. ω

C. -1

D. $-\omega$

7 Which of the following is a unit vector in the opposite direction to $-\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$?

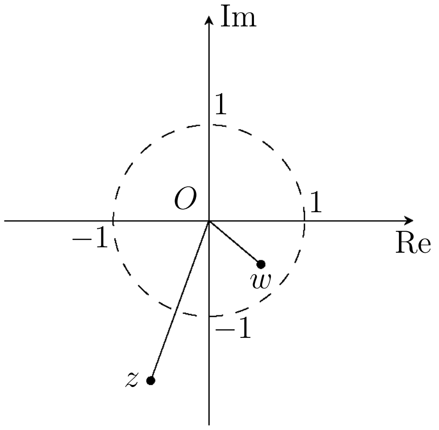
A. $\frac{1}{3}(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k})$

B. $\frac{1}{3}(-\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$

C. $\frac{1}{\sqrt{5}}\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$

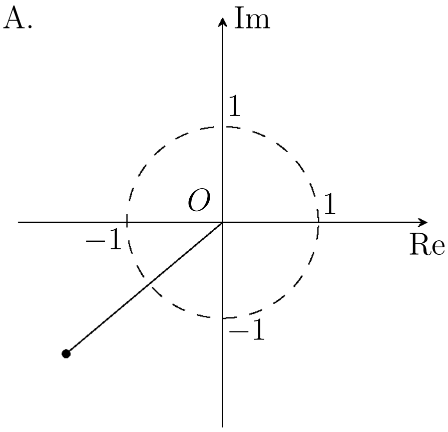
D. $\frac{1}{\sqrt{5}}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

8 The diagram shows the complex numbers z and w on the Argand diagram. The dotted circle is the unit circle.

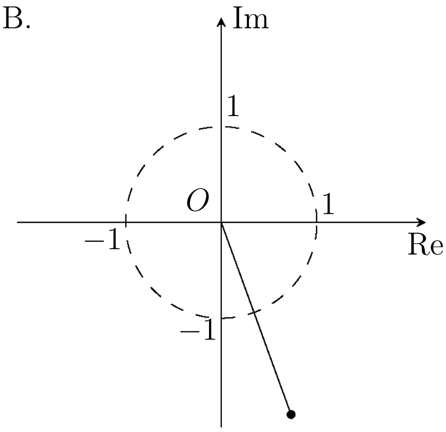


Which of the following best shows the position of $\dfrac{z}{w}$?

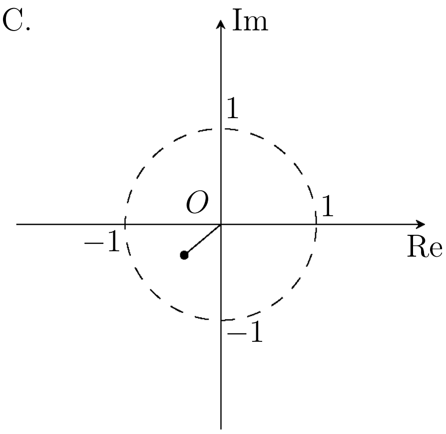
A.



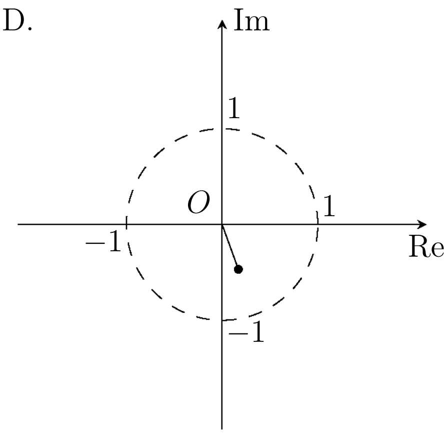
B.



C.



D.



9 Consider the following claim:

'If p is not prime then $(p-1)! + 1$ is not divisible by p'.

Which of the following is a counterexample to this claim?

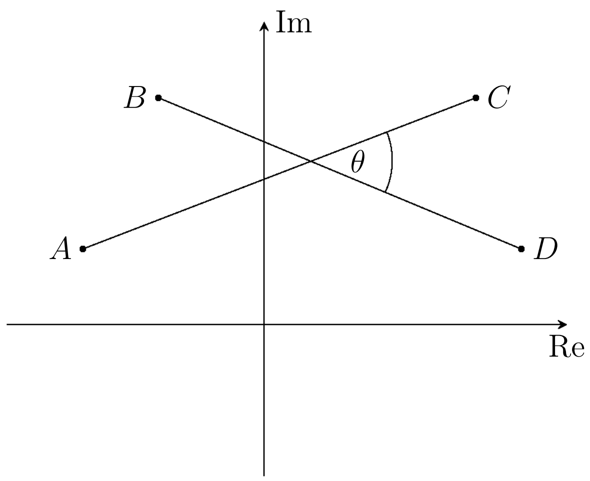
A. A prime number p such that $(p-1)! + 1$ is divisible by p.

B. A prime number p such that $(p-1)! + 1$ is not divisible by p.

C. A composite number p such that $(p-1)! + 1$ is divisible by p.

D. A composite number p such that $(p-1)! + 1$ is not divisible by p.

10 The points A, B, C and D on the Argand diagram represent the complex numbers a, b, c and d respectively. The angle θ is the acute angle between AC and BD. Which of the following is a correct expression for θ?



A. $\theta = \operatorname{Arg}\left(\frac{a-b}{c-d}\right)$

B. $\theta = \operatorname{Arg}\left(\frac{c-d}{a-b}\right)$

C. $\theta = \operatorname{Arg}\left(\frac{c-a}{d-b}\right)$

D. $\theta = \operatorname{Arg}\left(\frac{d-b}{c-a}\right)$

Section II

36 marks
Attempt Questions 11–13

Start each question on a NEW sheet of paper.

Question 11 (12 marks) Use a NEW sheet of paper.

(a) Let $z = 1 - \sqrt{3}\,i$.

(i) Express z in the form $re^{i\theta}$. **2**

(ii) Hence find z^{12} in simplest Cartesian form. **1**

(b) Find $\displaystyle\int \sin^{-1} x\, dx$. **2**

(c) Find $\displaystyle\int \frac{x^4 + x^2 + 1}{x^2 + 1}\, dx$. **2**

(d) Recall the standard integral $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C$. **2**

Using this standard integral, find $\displaystyle\int \sqrt{\frac{x+1}{x-1}}\, dx$.

(e) A particle of mass 5 kg moves in a straight line subject to a force F. **3**

The particle was initially at rest at the origin.

If $F = 15 - 10e^{-t}$, determine the displacement of the particle at time t.

End of Question 11

Question 12 (12 marks) Use a NEW sheet of paper.

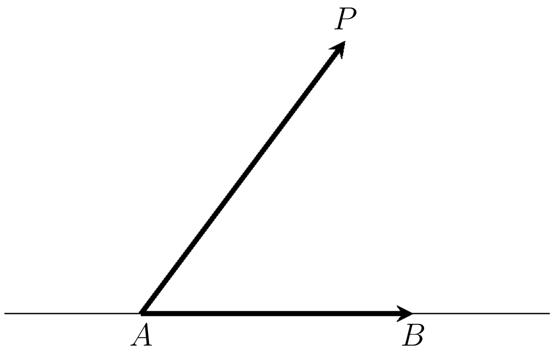
(a) (i) Find the square roots of $-18i$. **2**

(ii) Hence, solve $z^2 - (3+5i)z - 4 + 12i = 0$. **2**

(b) Sketch the region on the Argand diagram satisfied by $\text{Re}\left(\frac{1}{z}\right) \leqslant 1$. **3**

(c) Consider the points $A(2, -2, 0)$, $B(1, 5, 2)$ and $P(3, 3, 1)$. **2**

Find the projection of $\overrightarrow{AP}$ onto $\overrightarrow{AB}$ in simplest form.



(d) In the set of real numbers, let P be the proposition:

'If $a + b$ is irrational then at least one of a and b is irrational.'

(i) State the contrapositive of the proposition P. **1**

(ii) Write the converse of the proposition P, and state, with reasons, whether this converse is true or false. **2**

End of Question 12

Question 13 (12 marks) Use a NEW sheet of paper.

(a) Using the substitution $t = \tan\dfrac{x}{2}$, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{1+\sin x}\,dx$. **3**

(b) Consider the recurrence relation defined by $T_1 = 1$ and **3**

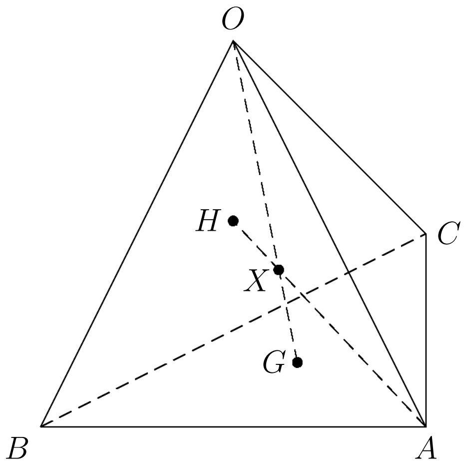
$$T_{n+1} = 3T_n + 4, \text{ for } n = 1, 2, \ldots.$$

Prove by mathematical induction that

$$T_n = 3^n - 2, \text{ for } n = 1, 2, \ldots.$$

(c) Prove that if $n \geqslant 3$ is an odd integer then $n^3 - n$ is divisible by 12. **3**

(d) Let $ABCD$ be a tetrahedron with $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. **3**



The points G and H satisfy $\overrightarrow{OG} = \dfrac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{3}$ and $\overrightarrow{OH} = \dfrac{\underset{\sim}{b}+\underset{\sim}{c}}{3}$.

Prove that OG and AH intersect at X, where $\overrightarrow{OX} = \dfrac{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}}{4}$.

End of task

Multiple choice

Q1	A
Q2	C
Q3	B
Q4	C
Q5	D
Q6	C
Q7	A
Q8	B
Q9	C
Q10	C

Q1

$$\int \frac{-1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\cos^{-1}3x + C \quad \therefore \boxed{(A)}$$

Q2 not (A and B) = (not A) or (not B)

$\therefore$ ~~n is not~~ divisible by 4 OR n is divisible by 8

$$\boxed{\therefore (C)}$$

Q3

$$\ddot{x} = -\frac{1}{v}$$

$$v\frac{dv}{dx} = -\frac{1}{v}$$

$$\frac{dv}{dx} = -\frac{1}{v^2} \qquad \boxed{\therefore (B)}$$

$$\frac{dx}{dv} = -v^2$$

$$x = C - \frac{v^3}{3}$$

Q4

$u = \sqrt{x}$

When $x = 0$, $u = 0$

When $x = 4$, $u = 2$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$$

$\therefore dx = 2u\,du$

$$\therefore \int_0^4 \frac{1}{x+\sqrt{x}}\,dx = \int_0^2 \frac{2u\,du}{u^2+u}$$

$\boxed{\therefore (C)}$

Q5

$P(z) = (z^2+1)Q(x) + 6z - 6i$

$\therefore P(i) = (i^2+1)Q(i) + 6i - 6i$

$= 0$

$\therefore 2i^3 + 3i(i^2) + ki - 3i = 0$

$-2 - 3 + k - 3 = 0$

$k = 8$

$\boxed{\therefore (D)}$

Q6

Since $\omega \neq 1$ is a cube root of unity

$$\omega^3 = 1 \Rightarrow (\omega - 1)(\omega^2 + \omega + 1) = 0$$

$$\Rightarrow 1 + \omega + \omega^2 = 0$$

$$\Rightarrow 1 + \omega^2 = -\omega$$

$$\therefore (1 + \omega^2)^3 = (-\omega)^3$$

$$= -\omega^3$$

$$= -1$$

$$\boxed{\therefore (C)}$$

Q7

$$\left| -\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k} \right| = \sqrt{(-1)^2 + 2^2 + (-2)^2}$$

$$= \sqrt{1 + 4 + 4}$$

$$= \sqrt{9}$$

$$= 3$$

$\therefore$ Unit vector in opposite direction to $-\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

is $-\frac{1}{3}(-\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}) = \frac{1}{3}(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k})$

$$\boxed{\therefore (A)}$$

Q8 Note that $|w| < 1$ and $|z| > 1$.

$\therefore \left|\frac{z}{w}\right| = \frac{|z|}{|w|} > 1$ $\therefore$ Outside unit circle

Also $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$ with $\mathrm{Arg}(z)$ and $\mathrm{Arg}(w)$ both negative. Hence

$$\mathrm{Arg}(z) < \mathrm{Arg}\left(\frac{z}{w}\right) < \mathrm{Arg}(w)$$

$$\boxed{\therefore (B)}$$

Q9 If P then Q is equivalent to

If not Q then not P.

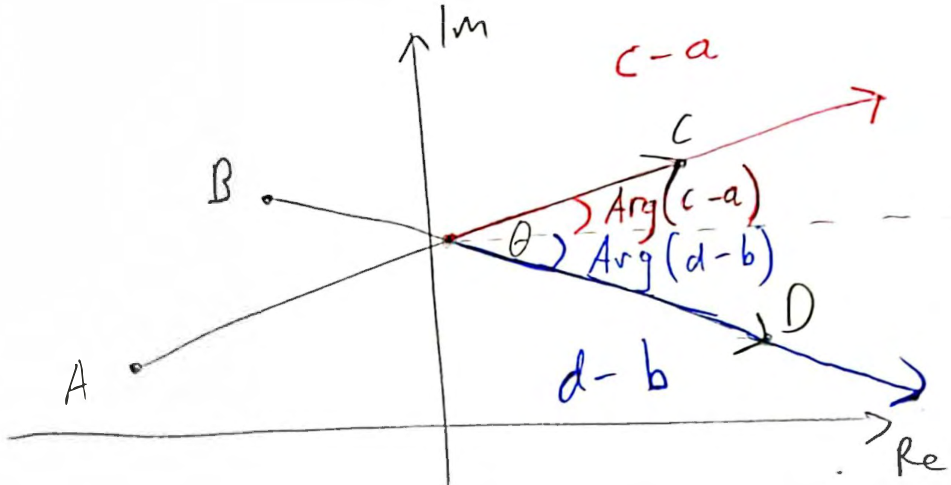
So the claim is equivalent to

"If $(p-1)! + 1$ is divisible by p, then p is prime".

A counterexample is therefore a not prime (composite) number such that $(p-1)! + 1$ is divisible by p.

$$\boxed{\therefore (C)}$$

Q10



$$\therefore \quad \theta = \mathrm{Arg}(c-a) - \mathrm{Arg}(d-b)$$

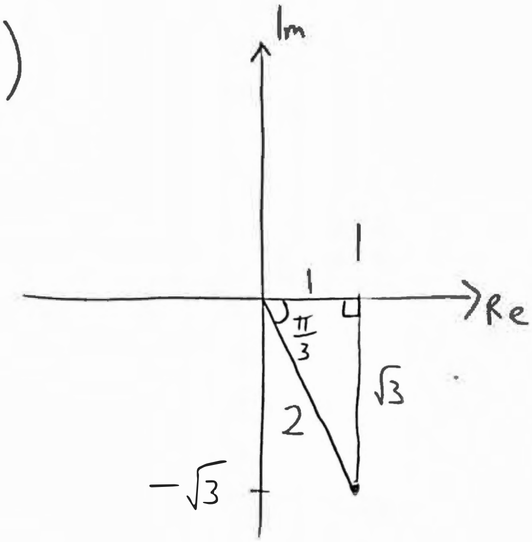
$$= \mathrm{Arg}\left(\frac{c-a}{d-b}\right)$$

[Note: $\mathrm{Arg}(d-b) < 0$]

$$\boxed{\therefore (c)}$$

Q11

(a) (i)



$z = 2e^{-\frac{i\pi}{3}}$ ✓ ✓ 2

- 1 mark for modulus
- 1 mark for argument

(ii) $z^{12} = \left(2e^{-\frac{i\pi}{3}}\right)^{12}$ 1

$= 2^{12} e^{-4i\pi}$

$= 4096$ ✓

(b) $\int \underbrace{\sin^{-1} x}_{u} \times \underbrace{1}_{v'} \, dx = uv - \int u'v \, dx$ 2

$= x \sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}} \times x \, dx$ ✓

$= x \sin^{-1} x + \sqrt{1-x^2} + C$ ✓

Q11

(c) $\displaystyle\int \frac{x^4+x^2+1}{x^2+1}\,dx$ 2

$$= \int \frac{x^2(x^2+1)+1}{x^2+1}\,dx$$

$$= \int \left(\frac{x^2\cancel{(x^2+1)}}{\cancel{x^2+1}} + \frac{1}{x^2+1} \right) dx$$

$$= \int \left(x^2 + \frac{1}{x^2+1} \right) dx \quad ✓$$

$$= \frac{x^3}{3} + \tan^{-1}x + C \quad ✓$$

(d) $\displaystyle\int \sqrt{\frac{x+1}{x-1}}\,dx = \int \frac{x+1}{\sqrt{(x-1)(x+1)}}\,dx$ Q 11 2

$$= \int \left(\frac{x}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}} \right) dx$$

$$= \sqrt{x^2-1} + \ln\left(x + \sqrt{x^2-1}\right) + C$$

(e) $F = ma$ 3

$$5a = 15 - 10e^{-t}$$

$$a = 3 - 2e^{-t}$$

$$v = 3t + 2e^{-t} + C$$

When $t=0$, $v=0$. $\therefore$ $C = -2$

$$v = 3t + 2e^{-t} - 2$$

$$x = \frac{3t^2}{2} - 2e^{-t} - 2t + D$$

When $t=0$, $x=0$ $\therefore$ $D = 2$

$$\boxed{x = \frac{3t^2}{2} - 2e^{-t} - 2t + 2}$$

Q12

(a) (i) $(x+iy)^2 = -18i$

$(x^2-y^2) + 2xyi = -18i$

Equating real and imaginary parts

$x^2 - y^2 = 0$ (1)

$xy = -9$ (2)

This gives $x+iy = \underline{3-3i}$ and $\underline{-3+3i}$

2

(ii) $\Delta = b^2 - 4ac$

$= [-(3+5i)]^2 - 4(1)(-4+12i)$

$= 9 - 25 + 30i + 16 - 48i$

$= -18i$

2

$\therefore z = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$= \dfrac{3+5i \pm (3-3i)}{2}$

$= 3+i,\ 4i$

Q12

(b) $z \neq 0$. Let $z = x + iy$ 3

$$\mathrm{Re}\left(\frac{1}{z}\right) = \mathrm{Re}\left(\frac{1}{x+iy} \times \frac{x-iy}{x-iy}\right)$$

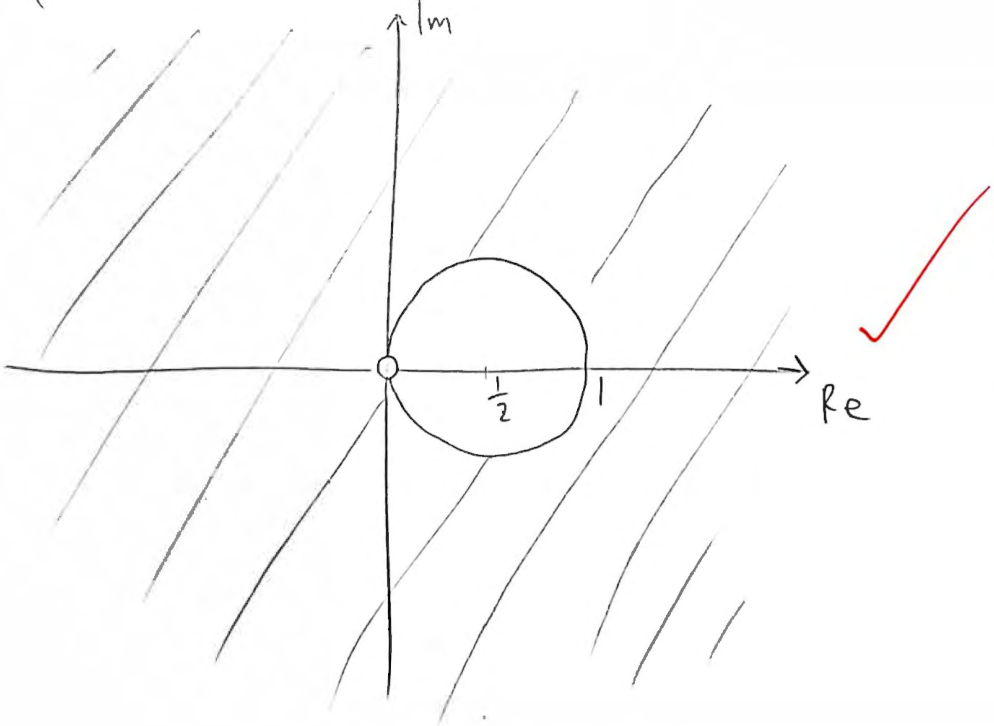
$$= \mathrm{Re}\left(\frac{x-iy}{x^2+y^2}\right)$$

$$= \frac{x}{x^2+y^2} \leq 1$$

$$x \leq x^2 + y^2$$

$$x^2 - x + y^2 \geqslant 0$$

$$\left(x - \frac{1}{2}\right)^2 + y^2 \geqslant \frac{1}{4}$$



- Open circle at the origin is required for full marks

Q12

(c) $\vec{AP} = \vec{OP} - \vec{OA} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$

2

$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$

$$\text{proj}_{\vec{AB}} \vec{AP} = \left(\frac{\vec{AB} \cdot \vec{AP}}{\vec{AB} \cdot \vec{AB}} \right) \vec{AB}$$

$$= \frac{\begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}}{\begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}} \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix} \checkmark$$

or equivalent correct expression for $\text{proj}_{\vec{AB}} \vec{AP}$

$$= \frac{-1 + 35 + 2}{1 + 49 + 4} \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

$$= \frac{36}{54} \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} \\ \frac{14}{3} \\ \frac{4}{3} \end{pmatrix} \checkmark$$

Q12

Contrapositive of P:

(d) (i) If a and b are rational, then $a+b$ is rational. 1

(ii) Converse of P: 2

If at least one of a and b is irrational, then $a+b$ is irrational.

This converse is false. For a counter-example, consider $a = \sqrt{2}$ and $b = -\sqrt{2}$. Then at least one of a and b is irrational, while $a+b = \sqrt{2} + (-\sqrt{2}) = 0$ is rational.

or equivalent counterexample

e.g. $a = \pi$, $b = -\pi$ e.t.c.

Q13

(a) Let $t = \tan\frac{x}{2}$ 　　When $x = 0$, $t = 0$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$ 　　When $x = \frac{\pi}{3}$, $t = \frac{1}{\sqrt{3}}$

$dt = \frac{1}{2}(t^2+1)\,dx$

$dx = \frac{2}{t^2+1}\,dt$ ✓

3

$$\therefore \int_0^{\frac{\pi}{3}} \frac{1}{1+\sin x}\,dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{t^2+1}\,dt$$

$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{t^2+1+2t}\,dt \checkmark$$

$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(t+1)^2}\,dt$$

$$= 2\left[-\frac{1}{t+1}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= 2\left(-\frac{1}{\frac{1}{\sqrt{3}}+1} + 1\right) \checkmark$$

$$= \sqrt{3} - 1 \quad \text{(after simplifying)}$$

simplifying fully not required for full marks

(b) ~~Base case:~~ If $n=1$,   Q13   3

$LHS = T_1$
$= 1$

$RHS = 3^1 - 2$
$= 1$
$= LHS$ ✓

Assume true for $n = k \geq 1$: $T_k = 3^k - 2$

Prove true for $n = k+1$:

R.T.P. $T_{k+1} = 3^{k+1} - 2$

$LHS = T_{k+1}$
$= 3T_k + 4$ ✓ (by definition)
$= 3(3^k - 2) + 4$ (by inductive assumption)
$= 3^{k+1} - 6 + 4$
$= 3^{k+1} - 2$
$= RHS$ ✓

$\therefore$ True by induction for $n = 1, 2, \ldots$

Q13

(c) $n^3 - n = n(n^2-1)$ 3

$= n(n-1)(n+1)$

$= (n-1)n(n+1)$

Since $n-1$, n and $n+1$ are three consecutive integers, one of them must be divisible by 3. ✓

$\therefore n^3 - n$ is divisible by 3.

Also, since n is odd we may write $n = 2m+1$, where $m \geq 1$ is an integer.

$\therefore n^3 - n = [(2m+1)-1](2m+1)[(2m+1)+1]$ ✓ sub in $n = 2m+1$

$= 2m(2m+1)(2m+2)$

$= 4m(2m+1)(m+1)$ ✓

which is divisible by 4.

$\therefore n^3 - n$ is divisible by 12, since it is divisible by 3 and 4.

(d) The line OG is given by: 3

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \overrightarrow{OG} = \lambda \left(\frac{\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}}{3} \right), \lambda \in \mathbb{R}.$$

The line AH is given by:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overrightarrow{OA} + \mu \overrightarrow{AH} = \underset{\sim}{a} + \mu \left(\frac{\underset{\sim}{b} + \underset{\sim}{c}}{3} - \underset{\sim}{a} \right), \mu \in \mathbb{R}.$$

These lines intersect when

$$\frac{\lambda}{3}\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}\right) = (1-\mu)\underset{\sim}{a} + \frac{\mu}{3}\underset{\sim}{b} + \frac{\mu}{3}\underset{\sim}{c}$$

Since $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ are linearly independent, we can equate coefficients:

$$\frac{\lambda}{3} = 1 - \mu \quad (1) \quad \text{AND} \quad \frac{\lambda}{3} = \frac{\mu}{3} \quad (2)$$

Sub (2) into (1): $\frac{\lambda}{3} = 1 - \lambda \implies \lambda = \mu = \frac{3}{4}$

Substitute $\lambda = \frac{3}{4}$ back into the line OG:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{\cancel{3}}{4}\left(\frac{\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}}{\cancel{3}} \right)$$

$$= \frac{\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}}{4},$$ as required.